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Robust Game Theoretic Synthesis in the Presence of Uncertain Initial States

Sinpyo Hong and Jason L. Speyer
University of California, Los Angeles, Los Angeles, California 90024

In this paper a performance robust control synthesis problem is considered; the performance measure ν represents the sensitivity of the output to the nonzero initial state. The problem is to find a controller, out of a prescribed set, that minimizes ν such that internal stability and a prescribed level of disturbance attenuation for all real parameter uncertainties in a given set are achieved. The primary motivation for considering this problem is that ν can be given a direct physical interpretation in terms of the transient response of a closed-loop system. Linear time-invariant systems with output feedback are considered. A game theoretic approach is employed to solve the synthesis problem and necessary conditions for optimality are given.

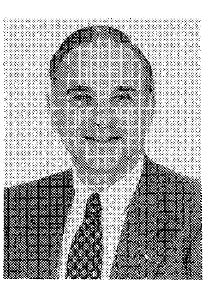
I. Introduction

MANY controller design problems are multicriteria problems, i.e., there is more than one performance criterion to be considered. For example, in the benchmark problem,[1] the controller is required to satisfy certain design specifications on the settling time, peak value of the control effort, and the sensitivity of the output to the measurement noise. Despite a significant research effort into the reduction of the sensitivity of the output to the process and measurement disturbances, for example, in $\mathcal{H}_\infty$ or linear quadratic Gaussian (LQG) control theories, improvement of transient performances measured as settling time and peak amplitude of output due to nonzero initial states draws little attention in control theory for linear time-invariant (LTI) systems. The usual LTI control problem has been concerned with optimization problems with a performance criterion on the sensitivity of output to the input process and measurement disturbances, with the result that various weighting strategies have been introduced to improve transient performances of the closed-loop system. However, only experienced designers have a good feel for the relation of the weight matrices to the transient behavior of closed-loop systems.

Strict performance requirements on the controller design often demand an exact model of the system to be controlled. However, uncertainties are usually involved in the mathematical description of physical systems. As a result, a closed-loop system in the presence of system uncertainty often has poor performance or even becomes unstable when the controllers are optimized with respect to a nominal model of the system. Hence, robustness of performance with respect to system uncertainty must also be considered in controller design. Many physical systems have real-valued structured uncertainty. For these systems, robust controller design based upon an unstructured uncertainty model, as in the $\mathcal{H}_\infty$ control, or on a complex-valued structured uncertainty model, as in μ-synthesis,[2] may produce conservative results.

These observations lead to the introduction of a constrained disturbance attenuation problem (CDAP). The idea in this problem can be roughly stated as follows: A controller is to be found that minimizes the worst-case transmission of initial state to the output out of a set of controllers that provide internal stability and achieve a prescribed level of disturbance attenuation for all real parameter uncertainties in a given set. The transmission of a nonzero initial state to the output is closely related to the transient response of the system. Hence, transient behavior of the control system as well as attenuation of disturbances are directly considered in CDAP, in contrast to the usual disturbance attenuation problem or $\mathcal{H}_\infty$ control problem in which only attenuation of disturbances is considered.

Sinpyo Hong was born in Pusan, Korea. He received a B.E. degree in 1982 from Pusan National University, a M.S. degree in 1985 from the Korea Advanced Institute of Science and Technology, and a Ph.D. degree in 1993 from the University of California, Los Angeles, all in Mechanical Engineering. Beginning in 1985 he was employed as an engineer at Korea Heavy Industry and Constructions (KHIC) Inc. where he worked in the Turbine Design Department. Following KHIC Inc., he was a research engineer at Samsung Electronics Inc., where he was involved in the development of a robot until 1990. Currently, he is a postdoctoral fellow at UCLA. His research interests include robust control with multicriteria.



Jason L. Speyer received the B.S. degree in aeronautics and astronautics from the Massachusetts Institute of Technology, Cambridge, MA, in 1960 and the Ph.D. degree in applied mathematics from Harvard University, Cambridge, MA, in 1968. His industrial experience includes Boeing, Raytheon, Analytical Mechanics Associated, and the Charles Stark Draper Laboratory. He was the Harry H. Power professor in Engineering and Engineering Mechanics, University of Texas, Austin. He is currently a Professor in the Mechanical, Aerospace, and Nuclear Engineering Department at the University of California, Los Angeles. Dr. Speyer is a Fellow of the American Institute of Aeronautics and Astronautics and the Institute of Electrical and Electronic Engineers. He was the 1985 recipient of the Mechanics and Control of Flight Award and 1995 Dryden Lectureship in Research.

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A game theoretic approach is employed for the design of the optimal controller to solve the CDAP. Initial states, process and measurement disturbances, and unknown system parameters are considered as the maximizing players, whereas the control is treated as the minimizing player in a minimax game problem. The game cost criterion consists of a positive term composed of a quadratic norm on the system output and negative terms composed of a quadratic norm on the disturbances. Instead of being penalized in the game cost criterion, initial state and uncertain system parameters are restricted to lie on or within the prescribed sets (here they are multidimensional ellipsoids). In contrast to the results in Ref. 3, the cost criterion is nonseparable and this formulation produces multiple worst initial states for a given linear controller. These differences make it difficult to use the simple method in Ref. 3 based upon two algebraic Riccati equations in solving the game problem. As a result, a linear fixed-order dynamic compensator based upon partial information is assumed. The controller is viewed as a function of a control parameter vector.

Throughout this paper $\|\cdot\|_A$ denotes the Euclidean norm weighted by A, $\|\cdot\|_2^2$ denotes $\int_0^\infty \|\cdot\|^2\,dt$, a superscript T denotes the transpose, and $\mathrm{Vec}(\cdot)$ denotes the Kronecker "Vec" operator. The vector is obtained by vertically stacking the columns of a matrix. The term $\lambda(\cdot)$ denotes an eigenvalue of a matrix, $\mathrm{co}\{\cdot\}$ denotes the convex hull of a set, $\bar{\lambda}(\cdot)$ denotes the largest eigenvalue of a matrix whose eigenvalues are real valued, and $\mathrm{Re}(\cdot)$ denotes the real part of a number. For a symmetric nonnegative matrix A the term $\sqrt{A}$ is the unique nonnegative symmetric matrix such that $\sqrt{A}\sqrt{A} = A$. Here, L_2 is used for $L_2[0, \infty)$. All vectors and matrices used in the following sections are real valued. For an algebraic Riccati equation (ARE)

$$A^T X + XA + XWX + Q = 0$$

with W, Q symmetric, the symmetric solution X is called a stabilizing solution if $A + WX$ is stable. The ARE has at most one stabilizing solution. A symmetric solution X' is called minimal if $X' \leq X$ for any nonnegative solution of the above ARE. If the stabilizing solution is nonnegative for nonnegative W and Q, then A is stable. All the proofs of lemmas and theorems of this paper are in the Appendix.

II. Problem Statement

Consider the dynamic system

$$\dot{x}(t) = A(p)x(t) + B(p)u(t) + \Gamma(p)w(t), \quad x(0) = x_0 \tag{1}$$

$$y(t) = C(p)x(t) + v(t) \tag{2}$$

and the associated input and output

$$w_i = \begin{bmatrix} \sqrt{W^{-1}}w \\ \sqrt{V^{-1}}v \end{bmatrix}, \qquad z = \begin{bmatrix} \sqrt{Q}x \\ \sqrt{R}u \end{bmatrix} \tag{3}$$

where $x(t) \in \boldsymbol{R}^n$ is the state, $u(t) \in \boldsymbol{R}^r$ is the control, $w(t) \in \boldsymbol{R}^m$ is the process disturbance, $y(t) \in \boldsymbol{R}^q$ is the measurement, $v(t) \in \boldsymbol{R}^q$ is the measurement disturbance, z is the output, w_i is the disturbance input, and $Q \geq 0$, $R > 0$, $W > 0$, $V > 0$. $p \in \boldsymbol{R}^l$ is the constant system parameter vector and the system matrices A, B, Γ, C are analytic functions, not necessarily linear functions, of p. Here, p is restricted to the set

$$E = \left\{p{:}\ \|p - \bar{p}\|^2_{\Sigma^{-2}} \leq 1\right\} \tag{4}$$

where $\bar{p}$ is the nominal value of p and $p - \bar{p}$ represents the uncertainty in p. The set E is given as a set associated with an ellipsoid although any compact set can be considered. The controller to be considered in this paper has the form

$$u = K_c\hat{x}, \qquad \dot{\hat{x}} = A_c\hat{x} + B_c y, \quad \hat{x}(0) = \hat{x}_0 \tag{5}$$

where $\hat{x} \in \boldsymbol{R}^s$. Hence, the control u is only the function of control parameter vector $K \in \boldsymbol{R}^k$ and $\hat{x}_0$, where $K = \mathrm{Vec}(K_c, A_c, B_c)$. Considering the controller structure, the system can be augmented such that

$$\dot{\bar{x}} = \bar{A}(p, K)\bar{x} + \bar{\Gamma}(p, K)\bar{w}, \quad \bar{x}(0) = \bar{x}_0 \tag{6}$$

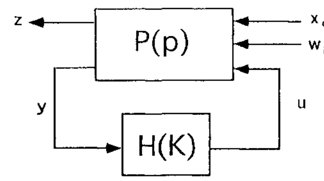

Fig. 1 Block diagram of system.

where $\bar{x}^T = [x^T, \hat{x}^T]$, $\bar{w}^T = [w^T, v^T]$, and

$$\bar{A}(p, K) = \begin{bmatrix} A(p) & B(p)K_c \\ B_cC(p) & A_c \end{bmatrix}, \quad \bar{\Gamma}(p, K) = \begin{bmatrix} \Gamma(p) & 0 \\ 0 & B_c \end{bmatrix} \tag{7}$$

Let $\mathcal{T}(p, K)$ be the transfer function between the output and the disturbance input such that $z = \mathcal{T}(p, K)w_i$. Then, for stable $\bar{A}(p, K)$,

$$\|\mathcal{T}(p, K)\|^2_\infty = \sup_{w_i} \frac{\|z\|_2^2}{\|w_i\|_2^2}, \quad \bar{x}_0 = 0, \qquad \|w_i\|_2^2 \neq 0 \tag{8}$$

We are interested in a class of controller with which the closed-loop system is stable and $\|\mathcal{T}(p, K)\|^2_\infty < \theta$ for all $p \in E$ for a given disturbance attenuation level θ.

Definition 2.1. Here, K is admissible if $\bar{A}(p, K)$ is asymptotically stable and $\|\mathcal{T}(p, K)\|^2_\infty < \theta$ for all $p \in E$. Let $E_A(\theta, E)$ denote the set of all admissible control parameter K. The term E_A is used for $E_A(\theta, E)$ whenever there is no confusion on θ and E.

A. Motivation

Consider Fig. 1, which represents the feedback system discussed above. $P(p)$ represents the open-loop system, which is a function of plant parameter p, and $H(K)$ represents the controller, which is a function of control parameter K. Given p and K with $\hat{x}_0 = 0$, the output z is determined by both disturbance input w_i and initial state x_0. The block diagram might be useful for the control system designers who are concerned with the effects of the nonzero initial state on the output, i.e., the transient behavior of the closed-loop system, as well as the effects of disturbance on the system performance. Note that the $\mathcal{H}_\infty$ norm of $\mathcal{T}(p, K)$ can be considered as the worst-case sensitivity of output to the disturbance input when the initial state is zero. One way of defining the worst-case sensitivity of output to the nonzero initial state without disturbance would be such that

$$\nu_0 = \max_{x_0} \frac{\|z\|_2^2}{\|x_0\|^2_{P_0^{-1}}}, \qquad w_i = 0, \qquad x_0 \neq 0 \tag{9}$$

which is introduced by Ghaoui et al.[4] Here, ν_0 can also be interpreted as the worst-case transmission of the initial state to the output. Since the two sensitivity norms $\|\mathcal{T}\|_\infty$ and ν_0 may conflict with each other, the control synthesis formulation based upon the minimization of either $\|\mathcal{T}\|_\infty$ or ν_0 may not be attractive to control system designers concerned with the effects both of disturbance and initial state on the output. One control synthesis formulation in which the two sensitivity norms are considered is a CDAP, which is described in the following. Consider the inequality

$$\|z\|_2^2 \leq \theta\|w_i\|_2^2 + \alpha\|x_0\|^2_{P_0^{-1}} \tag{10}$$

subject to Eqs. (1) and (2) with $K \in E_A$ and with minimizing $\hat{x}_0 = 0$. Define the set of α satisfying inequality (10) with given K, θ, E as

$$\mathcal{A}(K, \theta, E) = \big\{\alpha > 0{:}\ \text{Eq. (10) holds } \forall w, v \in L_2,$$
$$x_0 \in \boldsymbol{R}^n, p \in E\big\}$$

If $\alpha_1 \in \mathcal{A}$, then $\alpha_2 \in \mathcal{A}\ \forall \alpha_2 > \alpha_1$. Hence, only the infimum of $\mathcal{A}$ is meaningful. Hence the following definition is introduced:

$$\alpha_0(K, \theta, E) \triangleq \inf \mathcal{A}(K, \theta, E) \tag{11}$$

An important result concerning the disturbance attenuation that will be proved later is that if $K \in E_A$, then, for all $p \in E$,

$$\|\mathcal{T}(p, K)\|_\infty < \sqrt{\theta}, \qquad \nu_0(p, K) \leq \alpha_0(K, \theta, E)$$

This observation leads to the introduction of the CDAP: Given an upper bound on the $\mathcal{H}_\infty$ norm of $\mathcal{T}$, $\sqrt{\theta}$, and the set of real-valued uncertain system parameter, E, a control parameter vector K is to be found that minimizes $\alpha_0(K, \theta, E)$. Note that the CDAP is a performance robust control problem with a constraint on the attenuation level for the disturbance input w_i. The performance criterion to be minimized is directly related to the transient behavior of the closed-loop system. Let

$$\alpha_0^*(\theta, E) \triangleq \inf_{K \in E_A(\theta, E)} \alpha_0(K, \theta, E)$$

Then it is shown in Theorem 4.2 of Sec. IV that $\alpha_0^*(\theta, E)$ is a nonincreasing function of θ. Reduction in θ, the upper bound of $\|\mathcal{T}\|_\infty^2$, might be achieved by increasing α_0^*. This implies that θ and $\alpha_0^*(\theta, E)$ may form a Pareto optimal set. Hence, trade-off between two disturbance attenuation parameters θ and α_0 may be required in control synthesis. More discussions on the relationship between α_0 and transient performance are given in later sections. In the next section a game theoretic problem is formulated that is equivalent to the CDAP.

B. Robust Game Theoretic Problem

Consider the game cost criterion

$$J(K, \hat{x}_0, w, v, x_0, p) = \|z\|_2^2 - \theta\|w_i\|_2^2, \quad \theta > 0$$

subject to Eqs. (1–5) and the set constraining x_0 such that

$$D = \left\{x_0: \|x_0\|_{P_0^{-1}}^2 \leq 1\right\} \tag{12}$$

where $P_0 > 0$. The game problem is finding $\hat{x}_0 \in \boldsymbol{R}^s$ and $K \in E_A$ that are the solutions of the following minimax problem:

$$\min_{K, \hat{x}_0} \max_{p, x_0, w, v} J(K, \hat{x}_0, w, v, x_0, p)$$

where $p \in E$, $x_0 \in D$, and $w, v \in L_2$. In the following, the above dynamic game problem is transformed into a parameter minimax problem in which K is the minimizing parameter and p is the maximizing parameter, and the equivalence of CDAP to the game problem is shown.

The solution to the minimax problem involves the stabilizing solution to the ARE:

$$0 = \bar{A}(p, K)^T\Phi(p, K) + \Phi(p, K)\bar{A}(p, K) + (1/\theta)\Phi(p, K)\bar{\Gamma}(p, K)\bar{W}\bar{\Gamma}(p, K)^T\Phi(p, K) + \bar{Q}(K) \tag{13}$$

where $\bar{A}(p, K)$ and $\bar{\Gamma}(p, K)$ are defined in Eq. (7) and

$$\bar{Q}(K) = \begin{bmatrix} Q & 0 \\ 0 & K_c^T R K_c \end{bmatrix}, \qquad \bar{W} = \begin{bmatrix} W & 0 \\ 0 & V \end{bmatrix}$$

The corresponding Hamiltonian matrix is

$$H(p, K) = \begin{bmatrix} \bar{A}(p, K) & (1/\theta)\bar{\Gamma}(p, K)\bar{W}\bar{\Gamma}(p, K)^T \\ -\bar{Q}(K) & -\bar{A}(p, K)^T \end{bmatrix} \tag{14}$$

Before introducing main results of this section, a lemma is introduced that will be useful to characterize the admissible control parameter set E_A.

Lemma 2.1. Let $\bar{A}(p, K)$ in the ARE (13) be stable. Then the followings are equivalent[5–8]: 1) $\|\mathcal{T}(p, K)\|_\infty < \sqrt{\theta}$, 2) there exists a nonnegative stabilizing solution to the ARE (13), and 3) $H(p, K)$ has no eigenvalues on the imaginary axis.

Remark 2.1. Lemma 2.1 implies that E_A in Definition 2.1 is not a closed set.

Lemma 2.2. For $K \in E_A$ and $[w^{*T}, v^{*T}]^T = \bar{w}^* = \bar{W}\bar{\Gamma}(p, K)^T \Phi(p, K)\bar{x}/\theta$,

$$\max_{w, v \in L_2} J(K, \hat{x}_0, w, v, x_0, p) = J(K, \hat{x}_0, w^*, v^*, x_0, p) = \|\bar{x}_0\|_{\Phi(p,K)}^2$$

Proof. See Appendix.

Partition $\Phi(p, K)$ such that

$$\Phi(p, K) = \begin{bmatrix} \phi_{11}(p, K) & \phi_{12}(p, K) \\ \phi_{12}(p, K)^T & \phi_{22}(p, K) \end{bmatrix} \tag{15}$$

where $\phi_{12}(p, K) \in \boldsymbol{R}^{n \times s}$. Let

$$\nu_i(p, K) = \lambda_i(P_0\phi_{11}(p, K)), \quad \hat{x}_0^* = 0,$$

$$\nu(p, K) = \max_{i \in \hat{n}} \nu_i(p, K), \quad \hat{n} = \{1, 2, \ldots, n\}$$

$$\hat{n}_\nu(p, K) = \{i \in \hat{n}: \nu_i(p, K) = \nu(p, K)\}$$

$$\nu^*(K) = \max_{p \in E} \nu(p, K), \qquad \hat{p}_\nu(K) = \{p \in E: \nu(p, K) = \nu^*(K)\}$$

$$x_0^*(p, K) = \text{eigenvector of } P_0\phi_{11}(p, K) \quad \text{for} \quad \nu(p, K) \quad \text{with} \quad \|x_0^*(p, K)\|_{P_0^{-1}}^2 = 1 \tag{16}$$

Note that there may be more than one element in each of the sets $\hat{n}_\nu(p, K)$ and $\hat{p}_\nu(K)$.

Lemma 2.3. For $K \in E_A$ and $p^*(K) \in \hat{p}_\nu(K)$, we have

$$\min_{\hat{x}_0 \in R^s} \max_{p \in E} \max_{x_0 \in D} \|\bar{x}_0\|_{\Phi(p,K)}^2 = \left\| \begin{matrix} x_0^*(p^*(K), K) \\ \hat{x}_0^* \end{matrix} \right\|_{\Phi(p^*(K), K)}^2 = \nu^*(K)$$

Proof. See Appendix.

From Lemmas 2.2 and 2.3 we have

$$\min_{\hat{x} \in R^s} \max_{p \in E, x_0 \in D, w, v \in L_2} J(K, \hat{x}_0, w, v, x_0, p) = \nu^*(K)$$

Hence, the minimax game problem is equivalent to finding K that is the solution of the minimax problem

$$\min_{K \in E_A} \nu^*(K) = \min_{K \in E_A} \max_{p \in E} \nu(p, K)$$

The next theorem states the relationship between the minimax problem and the CDAP.

Theorem 2.1. For $K \in E_A \neq \emptyset$ and α_0 in Eq. (11),

$$\alpha_0(K, \theta, E) = \nu^*(K)$$

Proof. See Appendix.

Therefore, the equivalence of CDAP to the game problem follows from the equation

$$\inf_{K \in E_A} \alpha_0(K, \theta, E) = \inf_{K \in E_A} \nu^*(K) = \inf_{K \in E_A} \max_{p \in E} \nu(p, K)$$

III. Properties of CDAP

As mentioned in Sec. II.A, the CDAP is a control synthesis problem in which the transmission of two types of system input to the output are taken into consideration. One type of system input is the process and measurement disturbance, and the other one is the initial state. In contrast to the $\mathcal{H}_\infty$ optimal control theory in which only the minimization of the transmission of disturbance input energy to the output is concerned with assuming zero initial state, the CDAP is a constrained optimization problem in which a controller is to be found such that the transmission of initial state to the output is to be minimized with a constraint on the $\mathcal{H}_\infty$ norm, the transmission of unit disturbance input energy to the output. This section provides the relationship between the $\mathcal{H}_\infty$ optimal control problem and CDAP and some properties of CDAP.

Consider the Lyapunov equation

$$0 = L(p, K)\bar{A}(p, K) + \bar{A}(p, K)^T L(p, K) + \bar{Q}(K) \tag{17}$$

As $\Phi(p, K)$, partition $L(p, K)$ into four blocks that consist of $\Lambda_{11}(p, K)$, $\Lambda_{12}(p, K)$, and $\Lambda_{22}(p, K)$.

Lemma 3.1. If $\bar{A}(p, K)$ is stable, then $\nu_0(p, K) = \bar{\lambda}\,(P_0\Lambda_{11}(p, K))$, where $\nu_0(p, K)$ is defined in Eq. (9).[4]

Proof. See Appendix.

Lemma 3.2. For $K \in E_A$, $\nu_0(p, K) \le \nu(p, K)$. Hence, for $\alpha_0(K, \theta, E)$ defined in Eq. (11), we have

$$\nu_0(p, K) \le \nu(p, K) \le \nu^*(K) = \alpha_0(K, \theta, E) \quad \forall p \in E$$

Proof. See Appendix.

Considering the definition of E_A and Lemma 3.2, we have the following corollary.

Corollary 3.1. For $K \in E_A$ we have $\|\mathcal{T}(p, K)\|_\infty < \sqrt{\theta}$, $\nu_0(p, K) \le \nu^*(K)\ \forall p \in E$.

Note that for $\theta_2 > \theta_1$, $\mathcal{A}(K, \theta_1, E) \subset \mathcal{A}(K, \theta_2, E)$ and $E_A(\theta_1, E) \subset E_A(\theta_2, E)$. Note also that for $E_1 \subset E_2$, $\mathcal{A}(K, \theta, E_1) \supset \mathcal{A}(K, \theta, E_2)$ and $E_A(\theta, E_1) \supset E_A(\theta, E_2)$. Hence, we obtain the following theorems.

Theorem 3.1. Let $\theta_2 > \theta_1$. Then, $\alpha_0(K, \theta_1, E) \ge \alpha_0(K, \theta_2, E)$ $\forall K \in E_A(\theta_1, E)$. This implies

$$\inf_{K \in E_A(\theta_1, E)} \alpha_0(K, \theta_1, E) \ge \inf_{K \in E_A(\theta_2, E)} \alpha_0(K, \theta_2, E) \tag{18}$$

Theorem 3.2. Let $E_1 \subset E_2 \subset E$. Then, $\alpha_0(K, \theta, E_1) \le \alpha_0(K, \theta, E_2)\ \forall K \in E_A(\theta, E_2)$, and

$$\inf_{K \in E_A(\theta, E_1)} \alpha_0(K, \theta, E_1) \le \inf_{K \in E_A(\theta, E_2)} \alpha_0(K, \theta, E_2) \tag{19}$$

From Corollary 3.1, we can see that $\nu^*(K)$ might be used as a measure of the quality of transient performance of the system due to nonzero initial states with a controller parameter vector K; i.e., a small $\nu^*(K)$ means a small worst-case output of the system in response to the nonzero initial states while maintaining the disturbance attenuation level θ with respect to disturbances w and v. We can also see from Eq. (18) that a small θ, and hence a small $\mathcal{H}_\infty$ norm, is not necessarily desirable when a small value of α_0 is required. The inequality relation (19) implies that as the region of parameter uncertainties increases, the corresponding disturbance attenuation parameter α_0 increases. Therefore, the controller, which makes the dynamic system stable with the system parameter in a relatively large part of the allowable parameter region, has relatively poor worst-case transient performance due to nonzero initial states.

IV. Optimization Process

In this section, necessary conditions for the optimal K in the minimax game problem or CDAP are introduced. The synthesis procedure requires a solution to the parameter game problem

$$\min_{K \in E_A} \max_{p \in E} \nu(p, K) = \min_{K \in E_A} \nu^*(K) \tag{20}$$

Since the set E_A is not a closed set, and since $\nu^*(K)$ is not a convex function in general, there may not exist an optimal K that minimizes $\nu^*(K)$. Hence, in this section, a suboptimal problem is considered in which an optimal K is to be found in the closed subset of E_A. It is shown that the suboptimal problem is a special type of semi-infinite programming problem in which the function to be optimized and the functions in the inequality constraints are nondifferentiable. A general solution to this type of problem is not known yet. In Sec. IV.A, a necessary condition to the suboptimal problem is given for the case in which the functions in the constraints are locally Lipschitz continuous at the optimal K. This necessary condition involves generalized gradients for $\nu^*(K)$ and constraint functions. Hence, the condition is quite complicated to check numerically. In Sec. IV.B, an approximate form for the suboptimal problem and the corresponding necessary condition for optimality are introduced. The key feature of the approximated problem is that the semi-infinite programming problem can be transformed into the usual nonlinear programming problem with some mild assumptions on the differentiability of the functions in the suboptimal problem.

Without loss of generality we assume that $P_0 = I$. Otherwise, change such that $x \to \sqrt{P_0^{-1}}x$, $A(p) \to \sqrt{P_0^{-1}}A(p)\sqrt{P_0}$, $B(p) \to \sqrt{P_0^{-1}}B(p)$, $\Gamma(p) \to \sqrt{P_0^{-1}}\Gamma(p)$, $Q \to \sqrt{P_0}Q\sqrt{P_0}$, and $C(p) \to C(p)\sqrt{P_0}$. Since $\phi_{11}(p, K)$ is real symmetric, $\lambda_i(p, K)$ in Eq. (16) is continuously differentiable with respect to each element of $\Phi(p, K)$.[9] If $\Phi(p, K)$ is an analytic function of each element in $\bar{A}(p, K)$, $\bar{\Gamma}(p, K)$, and $\bar{Q}(K)$ in a subset of $E \times E_A$, then $\lambda_i(p, K)$ is continuously differentiable with respect to (p, K) in the set. The following lemma, which follows essentially from Lemma 1.1 in Ref. 10, shows that $\Phi(p, K)$ is an analytic function of each element in $\bar{A}(p, K)$, $\bar{\Gamma}(p, K)$, and $\bar{Q}(K)$ in $E \times E_A$.

Lemma 4.1. If K is admissible, then $\Phi(p, K)$ is an analytic function of $\bar{A}(p, K)$, $\bar{\Gamma}(p, K)$, and $\bar{Q}(K)$ for all $p \in E$.

Let $\{\mu_i(p, K): i \in \hat{n}_s, \hat{n}_s = \{1, 2, \dots, n+s\}\}$ be the set of all stable eigenvalues of $H(p, K)$, $d_{h_i}(p, K) = \mathrm{Re}(\mu_i(p, K))$, and $d_{a_i}(p, K) = \mathrm{Re}(\lambda_i(\bar{A}(p, K)))$. Then, using $\hat{n}_s$ instead of $\hat{n}$, we can define the functions and sets $(d_h(p, K), d_a(p, K))$, $(\hat{n}_h(p, K), \hat{n}_a(p, K))$, $(d_h^*(k), d_a^*(K))$, and $(\hat{p}_h(K), \hat{p}_a(K))$ in the same way as $\nu(p, K)$, $\hat{n}_\nu(p, K)$, $\nu^*(K)$, and $\hat{p}_\nu(K)$ are defined in Eq. (16), respectively. Then, using Lemma 2.1, it can be shown that

$$E_A = \{K \in \mathbf{R}^k\!: d_h(p, K) < 0,\ d_a(p, K) < 0\ \forall p \in E\}$$

Hence, the robust control synthesis problem (20) is equivalent to finding solution to the problem

$$\min_{K \in \mathbf{R}^k} \{\nu^*(K)\!: d_h(p, K) < 0, d_a(p, K) < 0\ \forall p \in E\} \tag{21}$$

Since E_A is not closed, there may not exist a minimum value of $\nu^*(K)$ in E_A. Hence, a minimization of $\nu^*(K)$ over a closed subset of E_A can be considered for practical purposes. Let

$$E_{A_c}(c_h, c_a) = \big\{K \in \mathbf{R}^k\!: d_h(p, K) + c_h \le 0,$$
$$d_a(p, K) + c_a \le 0 \quad \forall p \in E\big\}$$

where c_h, c_a are positive numbers. Then one modifications of the control synthesis problem could be finding a solution to $\min_{K \in E_{A_c}(c_h, c_a)} \nu^*(K)$. This problem is equivalent to

$$\min_{K \in \mathbf{R}^k} \Big\{\max_{p \in E} \nu(p, K)\!: d_h(p, K) + c_h \le 0,$$
$$d_a(p, K) + c_a \le 0 \quad \forall p \in E\Big\} \tag{22}$$

or

$$\min_{K \in \mathbf{R}^k} \{\nu^*(K)\!:\ d_h^*(K) + c_h \le 0,\ d_a^*(K) + c_a \le 0\} \tag{23}$$

This optimization problem is a special type of semi-infinite programming problem in which there are infinite number of constraints and the functions $\nu^*(K)$, $d_h(p, K)$, and $d_a(p, K)$ are not differentiable in general.

A. Necessary Condition for Optimal K

Since both $H(p, K)$ and $\bar{A}(p, K)$ are not symmetric matrices, eigenvalues of these matrices may not be differentiable with respect to (p, K) when these eigenvalues are defective, i.e., not diagonalizable.[9] Even if these matrices are nondefective, eigenvalues may not be locally Lipschitz continuous when they are not simple, i.e., when the algebraic multiplicity of each of the eigenvalues is greater than 1.[11] Hence, $d_h^*(K)$ and $d_a^*(K)$ in Eq. (23) can be the maximum function of functions that may not be locally Lipschitz continuous. General solution to this type of optimization problem is not known yet. In this section we are interested in finding a necessary condition for the problem given in Eq. (23) when the dominant eigenvalues of $H(p, K)$ and $\bar{A}(p, K)$ have some regularities. Note that a simple eigenvalue of a matrix is an analytic function of the elements of the matrix.[9,12] As can be seen in Eq. (23), the performance criterion ν and d_h and d_a in the constraint inequalities are nondifferentiable functions. Before the main results of this section are stated, some useful definitions for nonsmooth analysis found in the book by Clarke[13] are introduced.

Definition 4.1. A function f is locally Lipschitz continuous (l.L.c.) at $\hat{x}$ if there exist an $L \in [0, \infty)$, $\hat{\rho} > 0$, such that $\|f(x)$

$-f(x')\| \leq L\|x - x'\| \, \forall x, x' \in \mathcal{B}(\hat{x}, \hat{\rho})$, where $\mathcal{B}(\hat{x}, \hat{\rho})$ is the ball at center $\hat{x}$ with radius $\hat{\rho}$.

Definition 4.2. Let f be l.L.c. We define the (Clarke's) generalized directional derivative of $f(\cdot)$ at x in the direction h by

$$d_0 f(x\colon h) \triangleq \overline{\lim_{\substack{t \downarrow 0 \\ y \to x}}} \frac{f(y + th) - f(y)}{t}$$

where $\overline{\lim}(\cdot)$ is the limit superior of a function.

Definition 4.3. Let f be l.L.c. We define the (Clarke's) generalized gradient of $f(\cdot)$ at x by $\partial f \triangleq \{\xi\colon d_0 f(x\colon h) \geq \xi^T h \, \forall h\}$, where ξ and h are in the appropriate Euclidean space.

Definition 4.4. An l.L.c. function f is said to be regular if its (ordinary) directional derivative $[\mathrm{d}f(x\colon h)]$ exists for all x and h and $\mathrm{d}f(x\colon h) = \mathrm{d}_0 f(x\colon h)$.

We have the following lemma for the generalized gradient of $\nu^*(K)$.

Lemma 4.2. The generalized gradient of $\nu^*(K)$ is

$$\partial \nu^*(K) = \mathrm{co}\left\{ N^i = u_j^T \frac{\partial \phi_{11}(p, K)}{\partial K^i} u_j,\ j \in \hat{n}_\nu(p, K),\right.$$

$$\left. p \in \hat{p}_\nu(K), i \in \hat{k} \right\}$$

where $\hat{k} = \{1, 2, \ldots, k\}$, u_j is the orthonormal eigenvector of $\phi_{11}(p, K)$ corresponding to $\nu_j(p, K)$, and $N = [N^1 N^2 \cdots N^k]^T$, $K = [K^1 K^2 \cdots K^k]^T$.

Proof. See Appendix.

Let $\mathcal{L}(K, \eta_\nu, \eta_h, \eta_a) = \eta_\nu \nu^*(K) + \eta_h[d_h^*(K) + c_h] + \eta_a[d_a^*(K) + c_a]$, and $\partial_K \mathcal{L}(K, \eta_\nu, \eta_h, \eta_a) = \mathrm{co}\{\eta_\nu \partial \nu^*(K) + \eta_h \partial d_h^*(K) + \eta_a \partial d_a^*(K)\}$, where $\eta_h(\eta_a)$ is zero when $d_h^*(K) + c_h < 0$ $[d_a^*(K) + c_a < 0]$. Then, the following necessary condition for the optimal solution for Eq. (23) comes from Ref. 13, Theorem 6.1.1.

Theorem 4.1. Let K^* be the solution of Eq. (23). If $\partial d_h^*(K^*)$ and $\partial d_a^*(K^*)$ exist, then there exist $\eta_\nu \geq 0$, $\eta_h \geq 0$, and $\eta_a \geq 0$, not all zero, such that $0 \in \partial_K \mathcal{L}(K^*, \eta_\nu, \eta_h, \eta_a)$.

Remark 4.1. Since $d_{h_i}(p, K)$ is a continuous function of (p, K) for each $i \in \hat{n}_s$, $d_h(p, K)$ is a continuous function of (p, K). If $d_h(p, K)$ is l.L.c. for K uniformly in $p \in E$, then $d_h^*(K)$ is l.L.c., and hence the generalized gradient for $d_h^*(K)$ exists (Ref. 14, Theorem 3.1).

B. Approximate Solution

In the previous section the necessary condition for optimal K is quite complicated to check numerically. In this section functions that approximate the performance criterion and constraint functions are introduced. Using these approximate functions, a simple necessary condition for optimality and a simple search algorithm can be obtained under certain conditions on the differentiability of ν, d_h, and d_a. Let

$$L_{\nu_i}(p, K, \lambda_{\nu_i}) = \nu_i(p, K) + \lambda_{\nu_i}\left(\|p - \bar{p}\|^2_{\Sigma^{-2}} - 1\right) \tag{24}$$

$$L_\nu(p, K) = \max_{i \in \hat{n}} L_{\nu_i}(p, K, \lambda_{\nu_i}), \qquad L_\nu^*(K) = \max_{p \in E} L_\nu(p, K) \tag{25}$$

where the Lagrange multiplier λ_{ν_i} is zero when its associated constraint term $\|p - \bar{p}\|^2_{\Sigma^{-2}} - 1$ is not equal to zero. Note that if $\nu_i(p, K)$ is continuously differentiable with respect to (p, K), then $L_{\nu_i}(p, K, \lambda_{\nu_i})$ is also continuously differentiable with respect to (p, K). For $\epsilon_\nu > 0$, consider the following set of p:

$$\hat{p}_{\epsilon_\nu}(K) = \{p_\nu \in E\colon \nu^*(K) \geq \nu(p_\nu, K) > \nu^*(K) - \epsilon_\nu, \nu(p_\nu, K)$$
$$\text{is one of the local maximum of } \nu(p, K) \text{ for } p \in E\} \tag{26}$$

Then,

$$\frac{\partial}{\partial p} L_{\nu_i}(p, K, \lambda_{\nu_i}) = 0 \qquad \text{for} \qquad i \in \hat{n}_\nu \qquad \text{at} \qquad p \in \hat{p}_{\epsilon_\nu}(K) \tag{27}$$

Consider the following approximation function[15]

$$\bar{\nu}(K) = \frac{1}{\rho} \ell n \left(\sum_{p \in \hat{p}_{\epsilon_\nu}(K)} \exp[\rho L_\nu(p, K)] \right) \tag{28}$$

where $\rho > 0$. For finite $\hat{p}_{\epsilon_\nu}(K)$, we have

$$\nu^*(K) < \bar{\nu}(K) \leq \nu^*(K) + \left[\ell n N_{\epsilon_\nu}(K)\right]/\rho \tag{29}$$

where $N_{\epsilon_\nu}(K)$ is the number of elements in $\hat{p}_{\epsilon_\nu}(K)$. We can define the set of functions $L_{h_i}(p, K)$, $L_{a_i}(p, K)$, $L_h(p, K)$, $L_a(p, K)$, $L_h^*(K)$, $L_a^*(K)$, $\hat{p}_{\epsilon_h}(K)$, $\hat{p}_{\epsilon_a}(K)$, $\bar{d}_h(K)$, and $\bar{d}_a(K)$ in the same way as $L_{\nu_i}(p, K)$, $L_\nu(p, K)$, $L_\nu^*(K)$, $\hat{p}_{\epsilon_\nu}(K)$, and $\bar{\nu}(K)$ in Eqs. (24–28). Let N_{ϵ_h} and N_{ϵ_a} be the number of elements in $\hat{p}_{\epsilon_h}(K)$ and $\hat{p}_{\epsilon_a}(K)$, respectively. Then, using N_{ϵ_h} and N_{ϵ_a}, we can obtain the inequalities for $\bar{d}_h(K)$ and $\bar{d}_a(K)$, which are similar to Eq. (29).

Note that a global search over E is necessary to find $\hat{p}_{\epsilon_\nu}(K)$, $\hat{p}_{\epsilon_h}(K)$, and $\hat{p}_{\epsilon_a}(K)$ for each $K \in E_{A_c}(c_h, c_a)$. Suppose $\partial^2 L_\nu(p, K)/\partial p^2$ and $\partial^2 L_\nu(p, K)/(\partial K \partial p)$ exist, and suppose the Jacobian for $\partial^2 L_\nu(p, K)/\partial p^2$ is not zero for each $p \in \hat{p}_{\epsilon_\nu}(K)$. Then, by the implicit function theorem, all the elements in the set $\hat{p}_{\epsilon_\nu}(K)$ are differentiable functions of K. The same arguments can be applied for all the elements of $\hat{p}_{\epsilon_h}(K)$ and $\hat{p}_{\epsilon_a}(K)$. As an approximate form for the control synthesis problem (23), the following optimization problem might be considered:

$$\min_{K \in R^k} \{\bar{\nu}(K)\colon \bar{d}_h(K) + c_h \leq 0,\ \bar{d}_a(K) + c_a \leq 0\} \tag{30}$$

For this problem we have the following Kuhn and Tucker necessary condition:

Theorem 4.2. Let $\bar{K}$ be a solution to Eq. (30). Suppose that for each $p_\nu \in \hat{p}_{\epsilon_\nu}(\bar{K})$, $p_h \in \hat{p}_{\epsilon h}(\bar{K})$, and $p_a \in \hat{p}_{\epsilon a}(\bar{K})$, 1) $\partial^2 L_\nu(p_\nu, \bar{K})/\partial p^2$, $\partial^2 L_\nu(p_\nu, \bar{K})/(\partial K \partial p)$, $\partial^2 L_h(p_h, \bar{K})/\partial p^2$, $\partial^2 L_h(p_h, \bar{K})/(\partial K \partial p)$, $\partial^2 L_a(p_a, \bar{K})/\partial p^2$ and $\partial^2 L_a(p_a, \bar{K})/(\partial K \partial p)$ exist; 2) none of the Jacobians of $\partial^2 L_\nu(p_\nu, \bar{K})/\partial p^2$, $\partial^2 L_h(p_h, \bar{K})/\partial p^2$, and $\partial^2 L_a(p_a, \bar{K})/\partial p^2$ are zero; and 3) $N_{\epsilon_\nu}(\bar{K})$, $N_{\epsilon_h}(\bar{K})$, and $N_{\epsilon_a}(\bar{K})$ are finite.

Then, there exist $\beta_h \geq 0$ and $\beta_a \geq 0$ such that

$$\frac{\mathrm{d}}{\mathrm{d}K}\bar{\nu}(\bar{K}) + \beta_h \frac{\mathrm{d}}{\mathrm{d}K}\bar{d}_h(\bar{K}) + \beta_a \frac{\mathrm{d}}{\mathrm{d}K}\bar{d}_a(\bar{K}) = 0$$

where $\beta_h(\beta_a)$ is zero when $\bar{d}_h(\bar{K}) + c_h \neq 0$ $[\bar{d}_a(\bar{K}) + c_a \neq 0]$.

Remark 4.2. Even though eigenvalues of a symmetric matrix are continuously differentiable with respect to its elements,[12] they are, in general, not analytic functions of these elements. However, the maximum eigenvalue of $\phi_{11}(p, K)$ at $p \in \hat{p}_{\epsilon_\nu}(K)$ is usually a simple eigenvalue because it happens quite rarely that more than one eigenvalue of $\phi_{11}(p, K)$ assumes the same local maximum at a point in E when the volume of E is relatively large. Hence, it seems quite reasonable to assume that $\hat{n}_\nu(p, K)$ for each $p \in \hat{p}_{\epsilon_\nu}(K)$ has only one element and $\nu(p, K)$ is an analytic function for each $p \in \hat{p}_{\epsilon_\nu}(K)$.

Remark 4.3. The set of stable eigenvalues of $H(p, K)$ [and dominant eigenvalues of $\bar{A}(p, K)$] for each $p \in \hat{p}_{\epsilon_h}(K)[p \in \hat{p}_{\epsilon_a}(K)]$ usually consist of a pair of complex conjugate eigenvalues or a simple eigenvalues for the same reason as given in Remark 4.2. Hence, $L_h(p_h, K)$ and $L_a(p_a, K)$ are usually analytic functions at $p_h \in \hat{p}_{\epsilon_h}(K)$ and $p_a \in \hat{p}_{\epsilon_a}(K)$, respectively.

The optimization problem in Eq. (30) is the usual nonlinear programming problem. Let

$$\mathcal{L}(K) = \bar{\nu}(K) + \beta_h[\bar{d}_h(K) + c_h] + \beta_a[\bar{d}_a(K) + c_a]$$

If assumptions in Theorem 4.2 hold for a K that satisfies the constraints in (30), then the gradient of $\mathcal{L}(K)$ with respect to K is

$$\nabla \mathcal{L} = \frac{\mathrm{d}}{\mathrm{d}K}\bar{\nu}(K) + \beta_h \frac{\mathrm{d}}{\mathrm{d}K}\bar{d}_h(K) + \beta_a \frac{\mathrm{d}}{\mathrm{d}K}\bar{d}_a(K)$$

and from Eq. (27) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}K}\bar{\nu}(K) = \frac{\sum_{p\in\hat{p}_{\epsilon_\nu}(K)} \exp[\rho L_\nu(p,K)]\frac{\partial}{\partial K}\nu(p,K)}{\sum_{p\in\hat{p}_{\epsilon_\nu}(K)}\exp[\rho L_\nu(p,K)]}$$

We can obtain equations for $\mathrm{d}\bar{d}_h(K)/\mathrm{d}K$ and $\mathrm{d}\bar{d}_a(K)/\mathrm{d}K$ that are similar to the above equation. Any efficient numerical search algorithm can be used to find local optimal solution to (30) with the above gradient for $\mathcal{L}(K)$.

C. Role of c_a in Transient Performance

One of the most important characteristics of a control system is its transient response. Suppose there are no disturbances. Then given nonzero initial states, a desirable control system has a transient response with small amplitudes in the output but with a short settling time (the settling time can be roughly defined as the time required for the response curve to reach and stay within a given range about the final value, specified by an absolute percentage of the final value).[16] However, a controller that produces a small output for a nonzero initial state may not have as short a settling time as a controller that produces a large output. This observation necessitates in the control system design another performance criterion in addition to the quadratic performance index to handle settling time. Even though exact analytical expressions for settling time are prohibitively complicated for systems of order higher than 2 (Ref. 17) in classical control, settling time is estimated by the real part of the dominant closed-loop poles (the closed-loop poles that are the closest to the imaginary axis).

The real parts of the dominant closed-loop poles of the system are less than or equal to the real parts of the dominant eigenvalues of $\bar{A}(p, K)$. By definition, $-c_a$ is an upper bound for the real part of the eigenvalues of $\bar{A}(p, K)$ for all K in $E_{A_c}(c_h, c_a)$ and for all $p \in E$. Hence, if a short settling time is necessary for a control system, a large value of c_a is desirable.

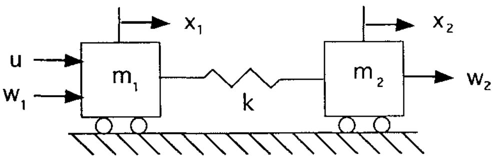

Fig. 2 Mass–spring system.

V. Application to Benchmark Problem

In this section, the game theoretic synthesis procedure is applied to the benchmark problem[1] of a two-mass–spring system. After introducing the benchmark problem (2), the design procedure for

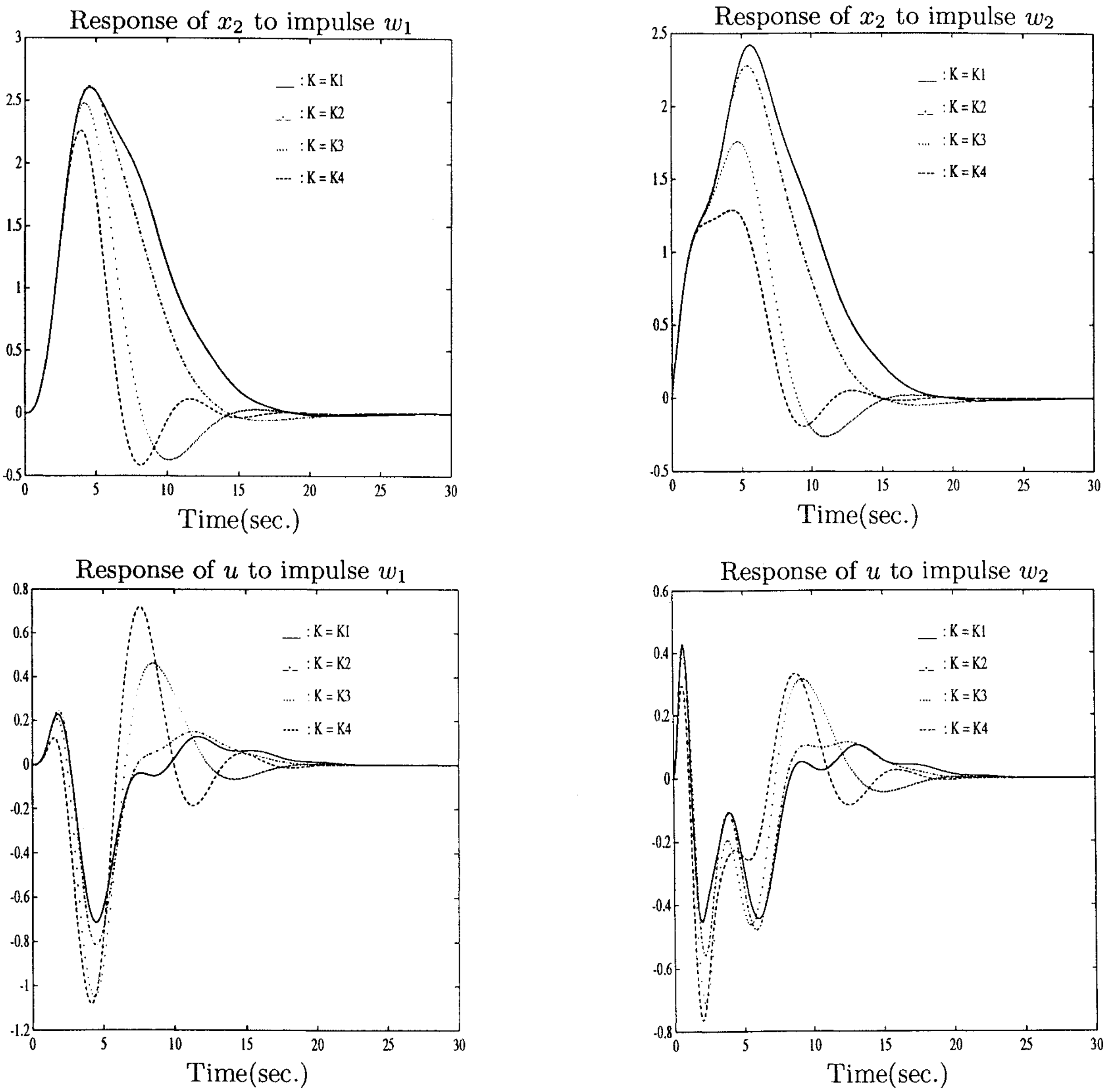

Fig. 3 Time responses of x_2 and u to unit impulse disturbances w_1 and w_2 with various K for nominal system parameters ($p = \bar{p}$). Figures show effects of d and ν on transient response of control system.

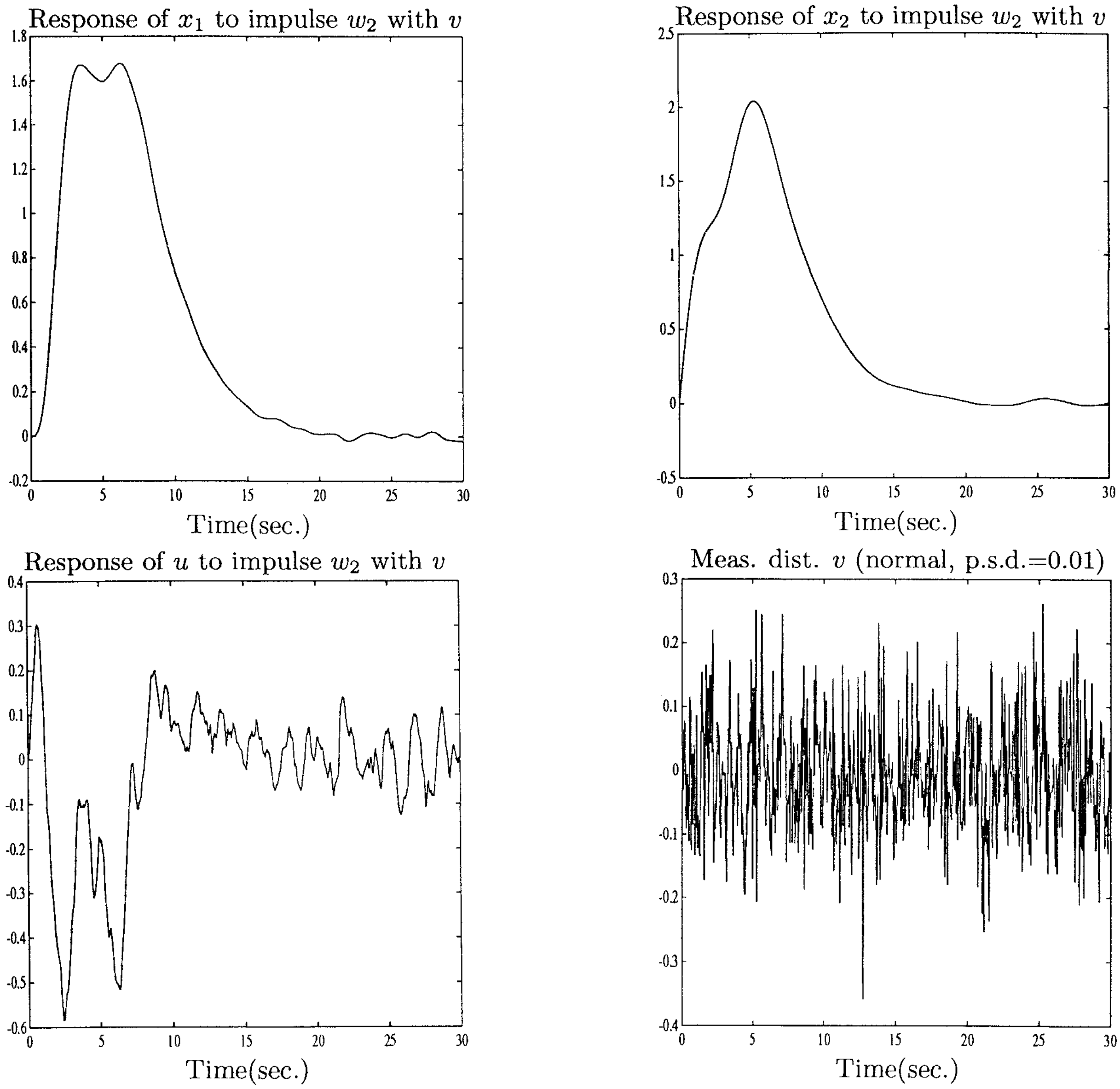

Fig. 4 Time responses of x_1, x_2, and u to unit impulse disturbance w_2 in presence of measurement disturbance v with K^* for nominal system parameter $(p = \bar{p})$.

the robust game theoretic controller derived in Sec. IV is presented to solve the benchmark problem.

A. Benchmark Problem

Consider the noncollocated mass–spring system in Fig. 2,[1] where the equation of motion is

$$\dot{x} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -k/m_1 & k/m_1 & 0 & 0 \\ k/m_2 & -k/m_2 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1/m_1 \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1/m_1 & 0 \\ 0 & 1/m_2 \end{bmatrix} w$$

$$y = [0 \quad 1 \quad 0 \quad 0]x + v, \qquad x = [x_1 \quad x_2 \quad x_3 \quad x_4]^T \qquad w = [w_1 \quad w_2]^T \tag{31}$$

where x is the state, u is the control, w is the process disturbance, y is the measurement, v is the measurement disturbance. Here, k, m_1, and m_2 are unknown system parameters and each of them has a nominal value of 1. The allowable ranges of the parameters are $0.5 \leq k \leq 2.0$, $0.5 \leq m_1 \leq 1.5$, and $0.5 \leq m_2 \leq 1.5$. It is desired to find a control u such that the closed-loop system is stable with relatively good transient performance for a relatively large region of system parameters in the above allowable set.

B. Controller Synthesis Procedure

To solve the benchmark problem (2), K_c, A_c, and B_c are selected such that

$$K_c = [K_5 \quad K_6 \quad K_7 \quad K_8], \qquad A_c = \begin{bmatrix} K_1 & K_2 & K_3 & K_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$B_c = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The control parameter vector is chosen such that $K = [K_1\, K_2\, K_3\, K_4\, K_5\, K_6\, K_7\, K_8]$. Let $\hat{x}_0 = 0$. The system unknown parameter vector and the nominal system parameter vector are selected as $p = [k\, m_1\, m_2]$, $\bar{p} = [1\ 1\ 1]$. We want to find the control u, which is a function of K, such that the system is stable with respect to the system parameter change in the allowable set while maintaining good impulse response to w_1 and w_2. Short settling time and small

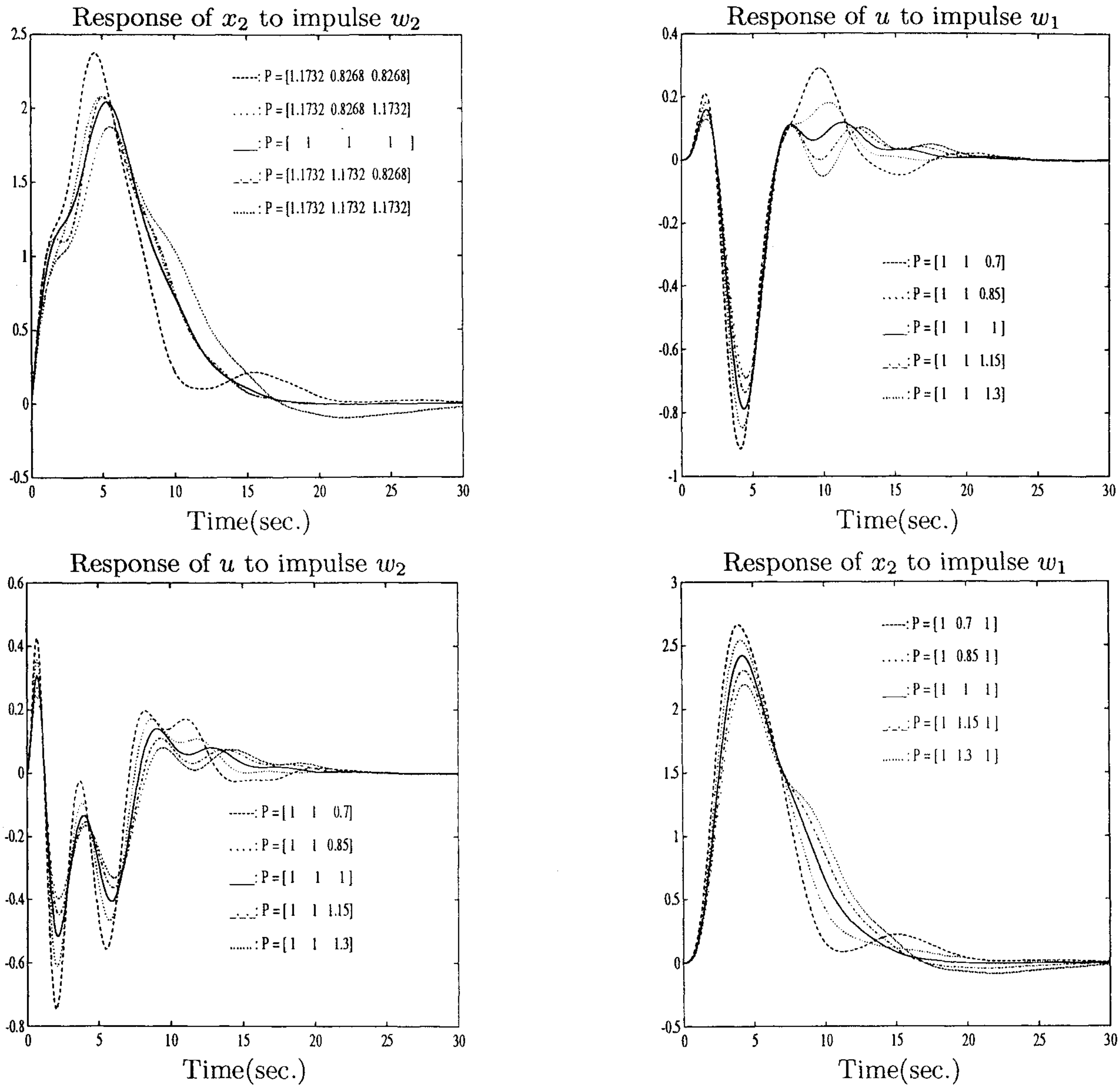

Fig. 5 Time responses of x_2 and u to unit impulse disturbances w_1 and w_2 with K^* for various system parameters in the E.

peak amplitude of the control are also required. Hence, we choose large θ and relatively small E so that good transient response is obtained. The design parameters are chosen as

$$\theta = 10^7, \qquad c_a = c_h = 0.16, \qquad Q = P_0 = I_4$$

$$W = I_2, \qquad V = R = 1$$

and $E = \{p\colon \|p - \bar{p}\|^2 \leq (0.3)^2\}$, where I_j denotes an identity matrix of dimension j. We find a control that satisfies the first-order necessary condition for K^* with a constraint using both global and local search methods. Although both global and local search can be used to determine $\hat{p}_v(K)$, $\hat{p}_h(K)$, and $\hat{p}_a(K)$, only local search is possible to find K^* because the allowable set for K is unknown. The global and local numerical search programs are written in Matlab. Gradient methods are used in the local search algorithm. In the global search for $\hat{p}_v(K)$, $\hat{p}_h(K)$, and $\hat{p}_a(K)$, the performance measures $v(p, K)$, $d_h(p, K)$, and $d_a(p, K)$ are tested at evenly distributed points in E with a given K. The worst points of p for each $v(p, K)$, $d_h(p, K)$, and $d_a(p, K)$ are selected among them. Starting from these points, $\hat{p}_v(K)$, $\hat{p}_h(p, K)$, and $\hat{p}_a(K)$ are determined with local search methods.

C. Solution and Simulation Results

Since $\theta \gg 1$, $d_h(p, K)$ is almost the same as $d_a(p, K)$. In the remainder of this section, $d(p, K)$ is used for both $d_h(p, K)$ and $d_a(p, K)$. To illustrate the relations of $v(p, K)$ and $d(p, K)$ to transient response, we fix p at $\bar{p}$ and change K such that v and d decrease. Figure 3 shows the responses of x_2 and u to unit impulses w_1 and w_2 with several values of K. Note that the peak amplitude of x_2 decreases as v decreases and settling time decreases as d decreases. Numerical values of K, v, and d are as follows:

$$K_1 = [-4.56 \quad -23.1 \quad -40.9 \quad -31.6 \quad 4.76 \quad 19.6 \quad -12.7 \quad -1.60]$$

$$K_2 = [-6.05 \quad -22.9 \quad -40.2 \quad -31.7 \quad 5.33 \quad 20.1 \quad -13.6 \quad -1.97]$$

$$K_3 = [-6.45 \quad -22.4 \quad -39.0 \quad -32.3 \quad 5.72 \quad 18.4 \quad -17.4 \quad -3.61]$$

$$K_4 = [-4.95 \quad -20.9 \quad -39.1 \quad -32.8 \quad 4.11 \quad 14.3 \quad -22.1 \quad -4.71]$$

$$d(\bar{p}, K1) = -0.28, \quad \nu(\bar{p}, K1) = 139$$
$$d(\bar{p}, K2) = -0.32, \quad \nu(\bar{p}, K2) = 114$$
$$d(\bar{p}, K3) = -0.40, \quad \nu(\bar{p}, K3) = 66$$
$$d(\bar{p}, K4) = -0.41, \quad \nu(\bar{p}, K4) = 43$$

To solve the benchmark problem (2),[1] a numerical search is performed to find a K that satisfies a necessary condition for the optimization problem (30). The resulting control parameter vector is

$$K^* = [-5.321 \quad -23.59 \quad -39.98 \quad -32.62 \quad 3.186 \quad 17.25 \quad -15.61 \quad -2.168]$$

or in transfer function form,

$$u(s) = 3.1859 \frac{(s+6.1888)(s-0.8966)(s+0.1226)}{(s^2+3.014s+14.3599)(s^2+2.3074s+2.2717)} y(s)$$

With K^* we have $\nu^*(K^*) = 123.8, d^*(K^*) = -c_h$, and $\|\mathcal{T}(\bar{p}, K^*)\|_\infty = 29.7$. The controller with K^* guarantees that $\nu(p, K^*)$ is smaller than or equal to 123.8 for all p in E. In addition, it is found by numerical test that K^* also guarantees that the closed-loop system is stable for all p in E with $\Sigma = 0.5I_3$. It is interesting to note that $\|\mathcal{T}(\bar{p}, K^*)\|_\infty$ is much smaller than its upper bound $\sqrt{\theta}$. It seems that c_a plays an important role to decrease the $\mathcal{H}_\infty$ norm. Figure 4 shows the time response of x_1, x_2, and u to unit impulse w_2 in the presence of measurement disturbance with nominal system parameter. In the simulation, white noise with zero mean and power spectral density of 0.01 is used as a measurement disturbance. It is interesting to note that the time response of x_2 is quite insensitive to the measurement noise even though the upper bound of the $\mathcal{H}_\infty$ norm of $\mathcal{T}$, $\sqrt{\theta}$, is relatively large. Figure 5 shows time responses of x_1, x_2, and u to unit impulses w_1 and w_2 with various system parameters in E. Note that the shapes of the time responses of x and u are changed relatively little with respect to system parameter changes in E. These figures illustrate the performance robustness of the game theoretic controller with respect to system parameter changes. In order to check the performance of a closed-loop system with K^* for the set $\{p: 0.5 \le k \le 2.0, 0.5 \le m_1 \le 1.5, 0.5 \le m_2 \le 1.5\}$, which is given in the benchmark problem 2, tests are made at the vertices of the set. Even though these points are out of the region E, the closed-loop system is stable at the points [0.5 0.5 1.5], [0.5 1.5 0.5], [2.0 0.5 1.5], [2.0 1.5 0.5], and [2.0 1.5 1.5] and unstable at the remaining three points. The settling times of the transient response to the impulse inputs at m_1 and m_2 are between 15 and 25 s, and the maximum controls for the same inputs are between 0.7 and 1.2 at each of the above five points except the points [2.0 1.5 1.5], in which the settling time is 35 s and the maximum control is 0.5.

Given the problem specifications,[1] numerical comparisons with other controllers for the benchmark problem in the September–October 1992 *Journal of Guidance, Control, and Dynamics* show that the present design performs better in terms of transient performance and does not suffer in any of the other measures.[18]

In order to find maxima of $\nu(p, K)$ and $d(p, K)$ over p in E, global searches over E are required. The optimum number of test points in E to find significant points for the global search depends on the problem and experience of the designer. In general, as the number of the test points increases, the chance to find all the significant points improves. However, the computational load for the search can increase exponentially as the dimension of E increases. Hence, for a system with a large number of uncertain parameters, an optimization technique based on upper bounds of the above functions over E can be an attractive alternative to achieve the trade-off between the computational load and conservative design. A state-space approach to controller design formulated with linear matrix inequalities appears to achieve this trade-off.

It is to be emphasized that identity matrices are used for the input and output weighting matrices. Hence, the iteration of changing of input and output weighting matrices, which is quite common in the usual control synthesis procedures, is not performed to improve transient performance in this application.

VI. Conclusions

A game theoretic synthesis is developed for the design of a LTI controller under real parameter uncertainties. In the formulation of the synthesis procedure, uncertainties such as the initial states, plant parameters, measurement and process disturbances are included as adversaries. Necessary conditions for the solution to the minimax problem are given. This synthesis procedure guarantees performance robustness as well as stability robustness for parameter variations belonging to a prescribed set. By considering the initial state uncertainties and dominant closed-loop poles, the resulting controller guarantees performance robustness with respect to initial state and system parameter variations. The performance criterion in the synthesis is closely related to the transient behavior of the closed-loop system. An interpretation of the game theoretic control synthesis in terms of the constrained disturbance attenuation problem clarifies the idea in the synthesis formulation.

The resulting suboptimal controller satisfying the necessary condition is applied to a benchmark problem. The simulation results demonstrate the performance robustness as well as stability robustness of a controller using this new synthesis method. The application to the benchmark problem illustrates the advantage of robust game theoretic synthesis over standard methods that require iterating the weighting matrix in the system input and output vectors. Although this is quite common in the usual control synthesis methods, the current synthesis technique does not require this complex heuristic iteration process to produce good transient performance of a closed-loop systems. In other words, for controller design using this control synthesis technique, we can choose relatively simple and suitable weighting matrices from physical intuition.

Appendix

Proof of Lemma 2.2. Consider the relation

$$J = \|\bar{x}_0\|^2_{\Phi(p,K)} - \|\bar{x}(\infty)\|^2_{\Phi(p,K)} - \theta \int_0^\infty \|\bar{w} - \bar{w}^*\|^2_{\bar{W}^{-1}} \, dt \quad \text{(A1)}$$

Since $K \in E_A$, $\bar{A}(p, K) + (1/\theta)\bar{\Gamma}(p, K)\bar{W}\bar{\Gamma}(p, K)^T\Phi(p, K)$ is asymptotically stable. Hence, $\bar{x}(\infty) = 0$ with $\bar{w}^* \in L_2$ and J assumes its maximum value with $\bar{w}^*$. □

Proof of Lemma 2.3. With $\hat{x}_0 = 0$,

$$\max_{p \in E, x_0 \in D} \|\bar{x}_0\|^2_{\Phi(p,K)} = \|x_0^*(p^*(K), K)\|_{\phi_{11}(p^*(K),K)} = \nu^*(K)$$

Hence,

$$\inf_{\hat{x}_0 \in \mathbf{R}^s} \max_{p \in E} \max_{x_0 \in D} \|\bar{x}_0\|^2_{\Phi(p,K)} \le \nu^*(K) \quad \text{(A2)}$$

We have $\|\bar{x}_0\|^2_{\Phi(p,K)} = x_0^T\phi_{11}(p, K)x_0 + 2x_0^T\phi_{12}(p, K)\hat{x}_0 + \hat{x}_0^T\phi_{22}(p, K)\hat{x}_0$. Note that if x_0 is an eigenvector of $P_0\phi_{11}(p, K)$, then $-x_0$ is also an eigenvector of $P_0\phi_{11}(p, K)$. Hence, for each $\hat{x}_0 \in \mathbf{R}^s$ and $p \in E$, we can find $x_0^*(p, K)$ such that $x_0^*(p, K)^T\phi_{12}(p, K)\hat{x}_0 \ge 0$. Hence, for each $\hat{x}_0 \in \mathbf{R}^s$

$$\max_{p \in E} \max_{x_0 \in D} \|\bar{x}_0\|^2_{\Phi(p,K)} \ge \nu^*(K)$$

This implies

$$\inf_{\hat{x}_0 \in \mathbf{R}^s} \max_{p \in E} \max_{x_0 \in D} \|\bar{x}_0\|^2_{\Phi(p,K)} \ge \nu^*(K) \quad \text{(A3)}$$

The proof of the lemma follows from Eqs. (A2) and (A3). □

Proof of Theorem 2.1. Considering Eq. (A1) for $\hat{x}_0 = 0$, we obtain

$$\|z\|_2^2 \le \theta\|w_i\|_2^2 + \|x_0\|^2_{\phi_{11}(p,K)} \ \forall w_i \in L_2 \quad \text{(A4)}$$

Noting that

$$\|x_0\|^2_{\phi_{11}(p,K)} \le \nu^*(K)\|x_0\|^2_{P_0^{-1}}, \qquad \forall x_0 \in \boldsymbol{R}^n \qquad p \in E$$

we have

$$\|z\|_2^2 \le \theta\|w_i\|_2^2 + \nu^*(K)\|x_0\|^2_{P_0^{-1}}, \qquad \forall w_i \in L_2$$

$$x_0 \in \boldsymbol{R}^n, \qquad p \in E$$

The proof of the theorem follows from the fact that equality holds in the above equation with $\bar{w} = \bar{w}^*$, $x_0 = x_0^*(p, K)$, and $p \in \hat{p}_\nu(K)$.

Proof of Lemma 3.1. For $w_i = 0$ and $\hat{x}_0 = 0$, we have $\|z\|_2^2 = \|x_0\|_{\Lambda_{11}(p,K)}$. For all $x_0 \in \boldsymbol{R}^n$, we have

$$\|x_0\|^2_{\Lambda_{11}(p,K)} \le \bar{\lambda}(P_0\Lambda_{11}(p, K))\|x_0\|^2_{P_0^{-1}} \tag{A5}$$

The proof follows from the fact that equality holds in Eq. (A5) when x_0 is the eigenvector of $P_0\Lambda_{11}(p, K)$ corresponding to $\bar{\lambda}[P_0\Lambda_{11}(p, K)]$. □

Proof of Lemma 3.2. Since Eq. (A4) holds with $w = v = 0$,

$$\|z\|_2^2 \le \|x_0\|^2_{\phi_{11}(p,K)} \le \nu(p, K)\|x_0\|^2_{P_0^{-1}} \qquad \forall x_0 \in \boldsymbol{R}^n$$

The proof follows from this relation and the definition of ν_0.

Proof of Lemma 4.2. Let $y = (p, u) \in \boldsymbol{R}^{l+n}$, $Y = E \times \{u \in C^n: \|u\| = 1\}$, and $M(y, K) = u^T\phi_{11}(p, K)u$. Then $\nu^*(K) = \max_{y\in Y} M(y, K)$. Let

$$\hat{Y}(K) = \{y \in Y: M(y, K) = \nu^*(K)\}$$

$$= \hat{p}_\nu(K) \times \{u: \phi_{11}(p, K)u = \nu(p, K)u, \|u\| = 1\}$$

Then, by Corollary 3.2,[14]

$$\partial\nu^*(K) = \mathrm{co}\left\{\frac{\partial}{\partial K}M(y, K), y \in \hat{Y}(K)\right\}$$

The proof of the theorem follows from the fact that $\partial M(y, K)/\partial K$ is a vector whose ith element is $u^T\partial\phi_{11}(p, K)u/\partial K^i$.

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